VANISHING AT INFINITY ON HOMOGENEOUS SPACES OF REDUCTIVE TYPE

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Abstract. Let G be a real reductive group and $Z = G/H$ a unimodular homogeneous G space. The space Z is said to satisfy VAI if all smooth vectors in the Banach representations $L^p(Z)$ vanish at infinity, $1 \leq p < \infty$. For H connected we show that Z satisfies VAI if and only if it is of reductive type.

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1. Introduction

In many applications of harmonic analysis of Lie groups it is important to study the decay of functions on the group. For example for a simple Lie group G, the fundamental discovery of Howe and Moore ([8], Thm. 5.1), that the matrix coefficients of non-trivial irreducible unitary representations vanish at infinity, is often seen to play an important role. In a more general context it is of interest to study matrix coefficients formed by a smooth vector and a distribution vector. If the distribution vector is fixed by some closed subgroup H of G, these generalized matrix coefficients will be smooth functions on the quotient manifold G/H. This leads to the question which is studied in the present paper, the decay of smooth functions on homogeneous spaces. More precisely, we are concerned with the decay of smooth L^p-functions on G/H.

Let G be a real Lie group and $H \subset G$ a closed subgroup. Consider the homogenous space $Z = G/H$ and assume that it is unimodular, that is, it carries a G-invariant measure μ_Z. Note that such a measure is unique up to a scalar multiple.

For a Banach representation (π, E) of G we denote by E^∞ the space of smooth vectors. In the special case of the left regular representation of G on $E = L^p(Z)$ with $1 \leq p < \infty$, it follows from the local Sobolev lemma that E^∞ is the space of smooth functions on Z, all of whose derivatives belong to $L^p(Z)$ (see [13], Thm. 5.1). Let $C_0^\infty(Z)$ be the space of smooth functions on Z that vanish at infinity. Motivated by the decay of eigenfunctions on symmetric spaces ([15]), the following definition was taken in [10]:

Definition 1.1. *We say Z has the property* VAI *(vanishing at infinity) if for all $1 \leq p < \infty$ we have*

$$L^p(Z)^\infty \subset C_0^\infty(Z).$$

By a result of [13], $Z = G$ has the VAI property for G unimodular and $H = \{\mathbf{1}\}$. The main result of [10] establishes that all reductive symmetric spaces admit VAI. On the other hand, it is easy to find examples of homogeneous spaces without this property. For example, it is clear that a non-compact homogeneous space with finite volume cannot have VAI.

Definition 1.2. *Let G be a real reductive group. We say that H is a* reductive subgroup *and that Z is of* reductive type *(or just reductive), if H is real reductive and the adjoint representation of H in the Lie algebra $\mathfrak{g}$ of G is completely reducible.*

Note that Z is unimodular in that case. The main result of this article is as follows.

Theorem 1.3. *Let G be a connected real reductive group and $H \subset G$ a closed connected subgroup such that $Z = G/H$ is unimodular. Then* VAI *holds for Z if and only if it is of reductive type.*

The direction 'if' is proved in Proposition 4.1. If Z is of reductive type and $B \subset G$ is a compact ball we provide essentially sharp lower and upper bounds for the volume $\mathrm{vol}_Z(Bz)$ with respect to μ_Z, where $z \in Z$ moves off to infinity (Section 3.2). These results generalize and simplify previous approaches in [11] and [9]. The lower bounds in particular imply that Z has VAI.

The converse implication is established in Proposition 5.1. The main lemma shows that in the non-reductive case the volume of the above mentioned sets Bz can be made exponentially small.

2. The invariant measure

In this section we provide a suitable framework for a discussion of the invariant measure on Z. Throughout G is a connected real reductive group and $H \subset G$ is a closed connected subgroup such that $Z := G/H$ is unimodular.

Let $\mathfrak{g}$ be the Lie algebra of G. We fix a Cartan involution θ of G. The derived involution $\mathfrak{g} \to \mathfrak{g}$ will also be called θ. The fixed point set of θ determines a maximal compact subgroup K of G whose Lie algebra will be denoted $\mathfrak{k}$. Let $\mathfrak{p}$ denote the -1-eigenspace of θ on $\mathfrak{g}$, then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let κ be a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$ such that

$$\kappa|_{\mathfrak{p}} > 0, \quad \kappa|_{\mathfrak{k}} < 0, \quad \mathfrak{k} \perp \mathfrak{p}.$$

Having chosen κ we define an inner product on $\mathfrak{g}$ by

$$\langle X, Y \rangle = -\kappa(\theta(X), Y).$$

We denote by $\mathfrak{h}$ the Lie algebra of H and by $\mathfrak{q}$ be its orthogonal complement in $\mathfrak{g}$.

Remark 2.1. Let Z be of reductive type. In this case the Cartan involution will be chosen such that it preserves H (see [7] Exercise VI A8 or [16] Thm. 12.1.4). It follows easily that then $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$. Moreover one has $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h}$ if and only if the pair $(\mathfrak{g}, \mathfrak{h})$ is symmetric, that is, if and only if

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \sigma(X) = X\}$$

for an involution σ of $\mathfrak{g}$. Then

$$\mathfrak{q} = \{X \in \mathfrak{g} \mid \sigma(X) = -X\}.$$

2.1. **Construction of the invariant measure.** The differential geometric way to obtain an invariant measure on Z is by defining an invariant differential form of top degree. Let us briefly recall this construction.

For every $g \in G$ we denote by

$$\tau_g : Z \to Z, \quad xH \mapsto gxH$$

the diffeomorphic left displacement by g on Z. Let $z_0 = H \in Z$ be the base point. Given $g \in G$ we shall identify the tangent space $T_{gz_0}Z$ of Z at the point gz_0 with $\mathfrak{g}/\mathfrak{h}$ via the map

$$\mathfrak{g}/\mathfrak{h} \to T_{gz_0}Z, \quad X + \mathfrak{h} \mapsto d\tau_g(z_0)X\,. \tag{2.1}$$

Let us emphasize that if $gz_0 = g'z_0$, then $g = g'h$ for some $h \in H$ and the two identifications differ by the automorphism $\mathrm{Ad}(h)|_{\mathfrak{g}/\mathfrak{h}}$. The assumption that an invariant measure exists on Z implies that the determinant of this automorphism is 1.

Let $Y_1, ..., Y_s$ be a basis of $\mathfrak{g}/\mathfrak{h}$ and $\omega_1, ..., \omega_s$ the corresponding dual basis in $(\mathfrak{g}/\mathfrak{h})^* \subset \mathfrak{g}^*$. We define the H-invariant volume form on $\mathfrak{g}/\mathfrak{h}$ by

$$\omega = \omega_1 \wedge \omega_2 \wedge ... \wedge \omega_s \in \bigwedge^s (\mathfrak{g}/\mathfrak{h})^*\,.$$

As ω is $\mathrm{Ad}(H)$-invariant we can extend ω to a G-invariant volume form ω_Z on Z. The measure μ_Z corresponding to ω_Z is then a Haar measure on Z.

3. Reductive Spaces

In this section $Z = G/H$ is of reductive type. Our goal is to establish uniform bounds for the invariant measure and deduce VAI for these spaces. As Z is of reductive type we can and will identify $\mathfrak{g}/\mathfrak{h}$ with $\mathfrak{q}$ in an H-equivariant way (see Remark 2.1). Note that $\mathfrak{q}$ is θ-stable and in particular $\mathfrak{q} = \mathfrak{q} \cap \mathfrak{k} + \mathfrak{q} \cap \mathfrak{p}$. We denote by $\mathrm{pr}_{\mathfrak{q}} : \mathfrak{g} \to \mathfrak{q}$ the orthogonal projection.

The characterization of infinity on Z is obtained by the polar decomposition which asserts that the polar map

$$\pi : K \times_{H\cap K} (\mathfrak{q} \cap \mathfrak{p}) \to Z, \quad [k, Y] \mapsto k\exp(Y)z_0 \tag{3.1}$$

is a homeomorphism (see [12] or [2], p. 74). Then a function $f \in C(Z)$ vanishes at infinity if and only if

$$\lim_{\substack{Y \mapsto \infty \\ Y \in \mathfrak{q} \cap \mathfrak{p}}} \sup_{k \in K} |f(\pi(k, Y))| = 0 .$$

3.1. **Local coordinates.** The objective of this subsection is to provide some useful local coordinates on Z and to give a uniform estimate of the invariant measure in terms of these local coordinates.

Let $U_R = \{X \in \mathfrak{g} : ||X|| < R\}$ for $R > 0$ and $U_{R,\mathfrak{q}} = U_R \cap \mathfrak{q}$. Note that when R is sufficiently small then $\exp|_{U_R}$ is diffeomorphic onto its image in G. We define for all $R > 0$ a 'ball' in G by $B_{R,\mathfrak{g}} = \exp(U_R)$. Likewise we define $B_{R,\mathfrak{q}} = \exp(U_{R,\mathfrak{q}}) \subset G$.

Let $g \in G$ and define a map ϕ_g by

$$\phi_g : U_{R,\mathfrak{q}} \to Z, \quad Y \mapsto \exp(Y) g z_0 .$$

Observe that

$$\mathrm{vol}_Z(B_{R,\mathfrak{g}} g z_0) \geq \mathrm{vol}_Z(B_{R,\mathfrak{q}} g z_0) = \int_{U_{R,\mathfrak{q}}} \phi_g^* \omega_Z$$

with the last equality holding if ϕ_g is diffeomorphic onto its image.

We will now show that ϕ_g is a coordinate chart with a Jacobian uniformly bounded from below provided $g = \exp(X)$ with $X \in \mathfrak{p} \cap \mathfrak{q}$ sufficiently large. We shall identify $T_{\exp(Y) g z_0} Z$ with $\mathfrak{q}$ as in (2.1). A standard computation yields for all $Y \in U_{\mathfrak{q},R}$:

$$\begin{aligned} d\phi_g(Y) : X \mapsto \mathrm{pr}_{\mathfrak{q}} \left(\mathrm{Ad}(g^{-1}) \left(\frac{1 - e^{-\mathrm{ad}\, Y}}{\mathrm{ad}\, Y} X \right) \right), \\ \mathfrak{q} \to T_{\exp(Y) g z_0} Z = \mathfrak{q} \end{aligned} \tag{3.2}$$

For $Y \in U_{\mathfrak{q},R}$ we shall denote by

$$J_g(Y) = |\det d\phi_g(Y)|$$

the Jacobian of ϕ_g at Y.

Lemma 3.1. *There exists a neighborhood $U \subset \mathfrak{q}$ of 0 and constants $C, d > 0$ such that*

$$J_{\exp(X)}(Y) \geq d \tag{3.3}$$

for all $X \in \mathfrak{p} \cap \mathfrak{q}$ with $\|X\| \geq C$, and all $Y \in U$. In particular, the map

$$\phi_{\exp X} : U \to Z$$

is then diffeomorphic onto its image.

Proof. We may assume that the basis $Y_1, \dots, Y_s$ of $\mathfrak{g}/\mathfrak{h}$ is an orthonormal basis of $\mathfrak{q}$. It then follows from (3.2) that

$$J_{\exp X}(Y) = |\det\left(\langle e^{-\operatorname{ad} X} \circ \left(\frac{1 - e^{-\operatorname{ad} Y}}{\operatorname{ad} Y}\right) Y_i, Y_j\rangle_{1\le i,j\le s}\right)|. \tag{3.4}$$

Since $\theta(X) = -X$ we can rewrite the matrix elements in (3.4) as

$$\langle\left(\frac{1 - e^{-\operatorname{ad} Y}}{\operatorname{ad} Y}\right) Y_i, e^{-\operatorname{ad} X} Y_j\rangle\,. \tag{3.5}$$

Observe that $\operatorname{ad} X$ is real semisimple and let $V_1, \dots, V_n$ be an orthonormal basis for $\mathfrak{g}$ of eigenvectors, with corresponding real eigenvalues $\lambda_1, ..., \lambda_n$. Let

$$b_{ik} = \langle\left(\frac{1 - e^{-\operatorname{ad} Y}}{\operatorname{ad} Y}\right) Y_i, V_k\rangle\,, \qquad c_{kj} = \langle V_k, Y_j\rangle,$$

then (3.5) equals $\sum_{k=1}^n b_{ik} c_{kj} e^{-\lambda_k}$. The determinant in (3.4) is a sum of products of such expressions.

We replace X by tX for $t \in \mathbb{R}$ and set

$$p(t) = p_{X,Y}(t) := \det\left(\langle\left(\frac{1 - e^{-\operatorname{ad} Y}}{\operatorname{ad} Y}\right) Y_i, e^{-\operatorname{ad} tX} Y_j\rangle_{1\le i,j\le s}\right).$$

Then it follows from the reasoning above that p is a linear combination of exponential functions $e^{-\lambda t}$ with exponents $\lambda \in \mathbb{R}$ which are sums of eigenvalues $-\lambda_k$. We observe that the exponents depend on X in a way which can be arranged to be locally uniform, and likewise for the dependence of the coefficients on X and Y.

For $Y = 0$ we have $p_{X,0}(-t) = p_{X,0}(t)$ for all t and all X, since $\theta(X) = -X$ and $\mathfrak{q}$ is θ-invariant. Thus $e^{\lambda t}$ and $e^{-\lambda t}$ will occur with the same coefficients in the expansion of $p_{X,0}$. If we denote by λ_X the maximal exponent λ such that $e^{\lambda t}$ occurs in $p_{X,0}(t)$ with a non-zero coefficient, then we conclude that $\lambda_X \ge 0$. By compactness and local uniformity it follows that there exists a compact neighborhood $U \subset \mathfrak{q}$ of 0 such that $e^{\lambda_X t}$ occurs in the expansion of $p_{X,Y}(t)$ with non-zero coefficient for all $Y \in U$ and any unit vector $X \in \mathfrak{p} \cap \mathfrak{q}$.

In the expansion of $p(t)$ the term with maximal exponent λ will dominate the others when $t \to \infty$. As we have just seen, this maximal exponent is ≥ 0 for all $Y \in U$ and all X. Hence there exist constants $C, d > 0$ such that $|p(t)| \ge d$ for $t > C$. Again by compactness, these constants can be chosen independently of $Y \in U$ and X with $\|X\| = 1$. $\square$

In general, the constant lower bound (3.3) is sharp. However, in many cases one can improve to an exponential lower bound. A particularly simple case is obtained when $(\mathfrak{g}, \mathfrak{h})$ is a symmetric pair. Let $\mathrm{pr}_s : \mathfrak{q} \to \mathfrak{q} \cap [\mathfrak{g}, \mathfrak{g}]$ denote the orthogonal projection of $\mathfrak{q}$ to its semisimple part, and let $X_s = \mathrm{pr}_s(X)$ for $X \in \mathfrak{q}$.

Lemma 3.2. *Assume G/H is a symmetric space. Then there exists a neighborhood $U \subset \mathfrak{q}$ of 0 and constants $C, d, \delta > 0$ such that*

$$J_{\exp(X)}(Y) \geq de^{\delta \|X_s\|}$$

for all $X \in \mathfrak{p} \cap \mathfrak{q}$ with $\|X\| \geq C$, and all $Y \in U$.

Proof. We denote by σ the involution of $\mathfrak{g}$ associated with $\mathfrak{h}$ (see Remark 2.1). Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ containing X and let Σ be the associated system of restricted roots. Root spaces in $\mathfrak{g}$ are denoted $\mathfrak{g}^\alpha$, where $\alpha \in \Sigma$. Let

$$\delta_X = \sum_{\alpha \in \Sigma, \alpha(X) > 0} \alpha(X)$$

(roots counted with multiplicities), then δ_X is independent of the choice of subspace $\mathfrak{a}$, and $\delta_X \geq \delta \|X_s\|$ for some constant $\delta > 0$, independent of X. We claim that there exist U, C and d as above such that

$$J_{\exp(X)}(Y) \geq de^{\delta_X} \tag{3.6}$$

for all $Y \in U$ and $\|X\| \geq C$. Obviously this will imply the lemma.

Notice that the $\mathrm{ad}\, X$-eigenspace for the eigenvalue $\lambda \in \mathbb{R}$ is given by $\mathfrak{g}_X^\lambda = \oplus_{\alpha \in \Sigma, \alpha(X) = \lambda} \mathfrak{g}^\alpha$. Note also that $V \in \mathfrak{g}_X^\lambda$ implies $\sigma(V) \in \mathfrak{g}_X^{-\lambda}$, since $\sigma(X) = -X$.

We follow the proof of Lemma 3.1. It suffices to prove that if λ_X is the maximal exponent in this proof then $\lambda_X t = \delta_{tX}$ for $t > 0$. It follows from the preceding paragraph that we can choose the orthonormal basis $(V_k)_{1,\dots,n}$ for $\mathfrak{g}$ such that basis vectors with non-zero eigenvalues $\pm\lambda$ for $\mathrm{ad}\, X$ are mutually paired by σ, and that each root with $\alpha(X) > 0$ is represented by such pairs according to its multiplicity. The orthonormal basis $(Y_j)_{1,\dots,s}$ for $\mathfrak{q}$ can then be chosen to consist of normalized multiples of the $V_k - \sigma(V_k)$ for each such pair, and additional vectors commuting with X. An elementary computation now shows that

$$p_{X,0}(t) = \Pi_{\alpha \in \Sigma, \alpha(X) > 0} \cosh(\alpha(tX)),$$

from which the expression for the maximal exponent follows. $\square$

3.2. **Volume bounds.** We record the following corollaries.

Corollary 3.3. *There exist a neighborhood $U \subset \mathfrak{q}$ of 0 and constants $C, d > 0$ such that the following holds for all $X \in \mathfrak{q} \cap \mathfrak{p}$ with $\|X\| > C$: For each $R > 0$ with $U_{R,\mathfrak{q}} \subset U$,*

$$\mathrm{vol}_Z(B_{R,\mathfrak{q}} \exp(X) z_0) \geq d \,\mathrm{vol}_{\mathfrak{q}}(U_{R,\mathfrak{q}}).$$

When G/H is symmetric, the lower volume bound can be improved to $de^{\delta\|X_s\|}\,\mathrm{vol}_{\mathfrak{q}}(U_{R,\mathfrak{q}})$ with $\delta > 0$ independent of X and R.

Remark 3.4. The proof of Lemma 3.1 also provides the following upper volume bound. There exist constants $D, \lambda > 0$ such that

$$\mathrm{vol}_Z(B_{R,\mathfrak{q}} \exp(X) z_0) \leq De^{\lambda\|X\|}\,\mathrm{vol}_{\mathfrak{q}}(U_{R,\mathfrak{q}})$$

for all $X \in \mathfrak{q} \cap \mathfrak{p}$. See [9] for such a bound in the literature.

Corollary 3.5. *There exist constants $C, d, R_0 > 0$ such that*

$$\mathrm{vol}_Z(B_{R,\mathfrak{g}} z) \geq dR^{\dim \mathfrak{q}}$$

for all $R \leq R_0$ and all $z = k \exp(X) z_0 \in Z$ with $k \in K$, $X \in \mathfrak{p} \cap \mathfrak{q}$ and $\|X\| > C$. Furthermore, in the symmetric case we have

$$\mathrm{vol}_Z(B_{R,\mathfrak{g}} z) \geq de^{\delta\|X_s\|} R^{\dim \mathfrak{q}}$$

with $\delta > 0$ independent of R and z.

Remark 3.6. For fixed $R > 0$ and G, H semisimple it was shown in [11] that there exists a constant $c > 0$ such that

$$\mathrm{vol}_Z(B_{R,\mathfrak{g}} z) > c$$

for all $z \in Z$. Corollary 3.5 sharpens this bound.

In the case of Z a semisimple symmetric space, the wave front lemma, Theorem 3.1, of [6], shows that for each $R > 0$ there exists an open neighborhood V of z_0 such that $B_{R,\mathfrak{g}} z$ contains a G-translate of V, for all $z \in Z$. This of course also implies that $\mathrm{vol}_Z(B_{R,\mathfrak{g}} z)$ is bounded below.

4. Reductive Spaces are VAI

Proposition 4.1. *Let Z be a homogeneous space of reductive type. Then*

$$L^p(Z)^\infty \subset C_0^\infty(Z)$$

for each $1 \leq p < \infty$, with continuous inclusion map. In particular, Z has the property VAI.

Proof. By applying the Sobolev inequality in local coordinates, we obtain the following for $1 \le p < \infty$ and for each compact neighborhood B of e in G (see [13] Lemma 5.1 for details). There exist finitely many elements $v_i \in \mathcal{U}(\mathfrak{g})$ (of degree up to the smallest integer $> \dim Z/p$) in the enveloping algebra of $\mathfrak{g}$, and for each $z \in Z$ a constant $D > 0$ such that

$$|f(z)| \le D \max \|(L_{v_i} f)\, \mathbf{1}_{Bz}\|_p \tag{4.1}$$

for all $f \in L^p(Z)^\infty$. Here $\mathbf{1}_{Bz}$ denotes the characteristic function of $Bz \subset Z$. The constant D is locally uniform with respect to z.

Based on Lemma 3.1, we can improve the local estimate (4.1), such that for G/H of reductive type it holds with D independent of z. Let $U \subset \mathfrak{q}$ and $C, d > 0$ be as in Lemma 3.1, and fix $R > 0$ such that $U_{R,\mathfrak{q}} \subset U$. It follows that

$$\|(f \circ \phi_{\exp X})\mathbf{1}_{U_{R,\mathfrak{q}}}\|_p \le d^{-1/p} \|f\, \mathbf{1}_{B_{R,\mathfrak{q}} \exp(X) z_0}\|_p \tag{4.2}$$

for $f \in L^p(Z)$ and $X \in \mathfrak{p} \cap \mathfrak{q}$ with $\|X\| > C$. By the Sobolev inequality for $\mathbb{R}^{\dim \mathfrak{q}}$, the value $|f \circ \phi_{\exp X}(0)|$ is estimated above by the p-norms, over any neighborhood of 0, of the derivatives of $f \circ \phi_{\exp X}$. Hence if $f \in L^p(Z)^\infty$ and $\|X\| > C$, we obtain an upper bound

$$|f(\exp(X) z_0)| \le D \max \|(L_v f)\, \mathbf{1}_{B_{R,\mathfrak{q}} \exp(X) z_0}\|_p$$

with derivatives as before by finitely many elements in $\mathcal{U}(\mathfrak{g})$, and with a constant D independent of f and X. After conjugation by $k \in K$ we conclude that (4.1) holds at $z = k \exp(X) z_0$, with $B = B_{R,\mathfrak{g}}$ and with a uniform constant D. As the set of elements $k \exp(X) z_0$ with $\|X\| \le C$ is compact, the inequality is finally obtained for all $z \in Z$.

The proposition is a straightforward consequence of the uniform version of (4.1). ☐

For G/H symmetric we obtain a stronger decay on the semisimple part of G by replacing the use of Lemma 3.1 by Lemma 3.2 in the estimate (4.2), while following the preceding proof:

Proposition 4.2. *Let $Z = G/H$ be symmetric. There exists a constant $\delta > 0$ with the following property. Let $f \in L^p(Z)^\infty$ where $1 \le p < \infty$. Then for each $\epsilon > 0$ there exists $C > 0$ such that*

$$|f(k \exp(X) z_0)| \le \epsilon e^{-\delta \|X_s\|}$$

for all $z = k \exp(X) z_0$, where $X \in \mathfrak{p} \cap \mathfrak{q}$, $\|X\| > C$ and $k \in K$.

5. Non-reductive spaces are not VAI

In this section we prove that VAI does not hold on any homogeneous space $Z = G/H$ of G, which is not of reductive type. We maintain the assumption that G is a connected real reductive group and establish the following result.

Proposition 5.1. *Assume that $H \subset G$ is a closed connected subgroup such that $Z = G/H$ is unimodular and not of reductive type. Then for all $1 \leq p < \infty$ there exists an unbounded function $f \in L^p(Z)^\infty$. In particular,* VAI *does not hold.*

The idea is to show that there is a compact ball $B \subset G$ and a sequence $(g_n)_{n\in\mathbb{N}}$ such that

- $Bg_nz_0 \cap Bg_mz_0 = \emptyset$ for $n \neq m$.
- $\mathrm{vol}_Z(Bg_nz_0) \leq e^{-n}$ for all $n \in \mathbb{N}$.

Out of these data it is straightforward to construct a smooth L^p-function which does not vanish at infinity.

Before we give a general proof we first discuss the case of unipotent subgroups. The argument in the general case, although more technical, will be modeled after that.

5.1. **Unipotent subgroups.** Let $H = N$ be a unipotent subgroup, that is, $\mathfrak{n} := \mathfrak{h}$ is an ad-nilpotent subalgebra of $[\mathfrak{g}, \mathfrak{g}]$. Now, the situation where $\mathfrak{n}$ is normalized by a particular semi-simple element is fairly straightforward and we shall begin with a discussion of that case.

If $X \in \mathfrak{g}$ is a real semi-simple element, i.e., $\mathrm{ad}\, X$ is semi-simple with real spectrum, then we denote by $\mathfrak{g}_X^\lambda \subset \mathfrak{g}$ its eigenspace for the eigenvalue $\lambda \in \mathbb{R}$, and by $\mathfrak{g}_X^\pm$ the sum of these eigenspaces for λ positive/negative. We record the triangular decomposition

$$\mathfrak{g} = \mathfrak{g}_X^+ + \mathfrak{z}_\mathfrak{g}(X) + \mathfrak{g}_X^- .$$

Here $\mathfrak{z}_\mathfrak{g}(X) =: \mathfrak{g}_X^0$ is the centralizer of X in $\mathfrak{g}$.

Lemma 5.2. *Assume that $\mathfrak{n}$ is normalized by a non-zero real semisimple element $X \in \mathfrak{g}$ such that $\mathfrak{n} \subset \mathfrak{g}_X^+$. Set $a_t := \exp(tX)$ for all $t \in \mathbb{R}$. Let $B \subset G$ be a compact ball around $\mathbf{1}$. Then there exists $c > 0$ and $\gamma > 0$ such that*

$$\mathrm{vol}_Z(Ba_tz_0) = c \cdot e^{t\gamma} \qquad (t \in \mathbb{R})$$

Proof. Let $A = \exp \mathbb{R}X$ and note that A normalizes N. Thus for all $a \in A$ the prescription

$$\mu_{Z,a}(Bz_0) := \mu_Z(Baz_0) \qquad (B \subset G \text{ measurable})$$

defines a G-invariant measure on Z. By the uniqueness of the Haar measure we obtain that

$$\mu_{Z,a} = J(a)\mu_Z$$

where $J : A \to \mathbb{R}_0^+$ is the group homomorphism $J(a) = \det \mathrm{Ad}(a)|_{\mathfrak{n}}$. The assertion follows. $\square$

Having obtained this volume bound we can proceed as follows. Let us denote by χ_k the characteristic function of $Ba_{-k}z_0 \subset Z$. We claim that the non-negative function

$$\chi := \sum_{k\in\mathbb{N}} k\chi_k \tag{5.1}$$

lies in $L^p(G/H)$. In fact

$$\|\chi\|_p \leq \sum_{k\in\mathbb{N}} k\|\chi_k\|_p \leq c\sum_{k\in\mathbb{N}} ke^{-\gamma k/p}\,.$$

Finally we have to smoothen χ: For that let $\phi \in C_c(G)^\infty$ with $\phi \geq 0$, $\int_G \phi = 1$ and $\operatorname{supp}\phi \subset B$. Then $\tilde{\chi} := \phi * \chi \in L^p(Z)^\infty$ with $\tilde{\chi}(a_{-k}z_0) \geq k$. Hence $\tilde{\chi}$ is unbounded.

In general, given a unipotent subalgebra $\mathfrak{n}$, there does not necessarily exist a semisimple element which normalizes $\mathfrak{n}$. For example if $U \in \mathfrak{g} = \mathfrak{sl}(5,\mathbb{C})$ is a principal nilpotent element, then $\mathfrak{n} = \mathrm{span}\{U, U^2 + U^3\}$ is a 2-dimensional abelian unipotent subalgebra which is not normalized by any semi-simple element of $\mathfrak{g}$. The next lemma offers a remedy out of this situation by finding an ideal $\mathfrak{n}_1 \lhd \mathfrak{n}$ which is normalized by a real semisimple element X with $\mathfrak{n} \subset \mathfrak{g}_X^+ + \mathfrak{g}_X^0$.

Lemma 5.3. *Let $\mathfrak{n} \subset [\mathfrak{g},\mathfrak{g}]$ be an* ad*-nilpotent subalgebra and let $0 \neq U \in \mathfrak{z}(\mathfrak{n})$. Then there exists a real semi-simple element $X \in \mathfrak{g}$ such that $[X,U] = 2U$ and $\mathfrak{n} \subset \mathfrak{g}_X^+ + \mathfrak{g}_X^0$.*

Proof. According to the Jacobson-Morozov theorem one finds elements $X, V \in \mathfrak{g}$ such that $\{X, U, V\}$ form an $\mathfrak{sl}_2$-triple, i.e. satisfy the commutator relations $[X,U] = 2U$, $[X,V] = -2V$, $[U,V] = X$. Note that $\mathfrak{n} \subset \mathfrak{z}_{\mathfrak{g}}(U)$ and that $\mathfrak{z}_{\mathfrak{g}}(U)$ is $\mathrm{ad}\,X$-stable. It is known and in fact easy to see that $\mathfrak{z}_{\mathfrak{g}}(U) \subset \mathfrak{g}_X^+ + \mathfrak{g}_X^0$. All assertions follow. $\square$

Within the notation of Lemma 5.3 we set $\mathfrak{n}_1 = \mathbb{R}U$ and $N_1 = \exp(\mathfrak{n}_1)$. Furthermore we set $Z_1 = G/N_1$ and consider the contractive averaging map

$$L^1(Z_1) \to L^1(Z), \quad f \mapsto \widehat{f}; \quad \widehat{f}(gN) = \int_{N/N_1} f(gnN_1)\; d(nN_1).$$

Let $B \subset G$ be a compact ball around $\mathbf{1}$, of sufficiently large size to be determined later, and let $B_1 = B \cdot B \subset G$. Let χ be the function on Z_1 constructed as in (5.1), using the element X from Lemma 5.3 and the compact set B_1. Let $\widehat{\chi} \in L^1(Z)$ be the average of χ. We claim that $\widehat{\chi}(Ba_{-k}z_0) \geq k$ for all k. In fact let $Q \subset N/N_1$ be a compact neighborhood of $\mathbf{1}$ in N/N_1 with $\mathrm{vol}_{N/N_1}(Q) = 1$. Then for B large enough we have $a_{-k}Qa_k \subset B$ for all k (Lemma 5.3). Hence for $b \in B$,

$$\widehat{\chi}(ba_{-k}z_0) \geq \int_Q \chi(ba_{-k}nN_1)\, d(nN_1) \geq k,$$

proving our claim.

To continue we conclude that $f_p := (\widehat{\chi})^{\frac{1}{p}} \in L^p(Z)$ is a function with $f_p(Ba_{-k}z_0) \geq k$ for all k. Finally we smoothen f_p as before and conclude that VAI does not hold true.

5.2. **The general case of a non-reductive unimodular space.** Finally we shall prove Proposition 5.1 in the general situation where H is a closed and connected subgroup for which $Z = G/H$ is unimodular and not of reductive type.

We fix a Levi-decomposition $\mathfrak{h} = \mathfrak{r} \rtimes \mathfrak{s}$ of $\mathfrak{h}$. As in Section 2 we fix a Cartan involution θ of $\mathfrak{g}$, and by Remark 2.1 we may assume that it restricts to a Cartan involution of $\mathfrak{s}$.

Proof of Proposition 5.1.

We will argue by induction on $\dim \mathfrak{g}$, the base of the induction being clear. We will perform a number of reduction steps (which may involve the induction hypothesis) that will lead us to a simplified situation which is described in Step 9 of the proof.

Step 1: $\mathfrak{h}$ is not contained in any reductive proper subalgebra of $\mathfrak{g}$.

Indeed, otherwise $\mathfrak{h}$ is contained in a proper subalgebra $\tilde{\mathfrak{h}}$ of $\mathfrak{g}$, which is reductive in $\mathfrak{g}$. Then $\mathfrak{h}$ is not reductive in $\tilde{\mathfrak{h}}$ ([4], §6.6 Cor. 2). By induction $\tilde{H}/H$ is not VAI, in the strong sense that for every $1 \leq p < \infty$ there exists an unbounded function $f \in L^p(\tilde{H}/H)^\infty$. We claim that G/H is not VAI in the same strong sense. Let $\tilde{\mathfrak{q}} \subset \mathfrak{g}$ be the orthogonal complement to $\tilde{\mathfrak{h}}$ in $\mathfrak{g}$. Then for a small neighborhood $V \subset \tilde{\mathfrak{q}}$ of 0 the tubular map

$$V \times \tilde{H} \to G, \quad (X, h) \mapsto \exp(X)h$$

is diffeomorphic. The Haar measure on G is expressed by $J(X)dXdh$ with $J > 0$ a bounded positive function. Since $\tilde{H}$ normalizes $\tilde{\mathfrak{q}}$, this allows us to extend smooth L^p-functions from $\tilde{H}/H$ to G/H and we see that G/H is not VAI in the strict sense. Hence we may assume as stated in Step 1.

Step 2: $\mathfrak{h}$ is contained in a maximal parabolic subalgebra $\mathfrak{p}_0$.

Indeed, by the characterization of maximal subalgebras of $\mathfrak{g}$ (see [5], Ch. 8, §10, Cor. 1), a maximal subalgebra is either a maximal parabolic subalgebra or it is a maximal reductive subalgebra. Hence it follows from Step 1 that $\mathfrak{h}$ is contained in a maximal parabolic subalgebra $\mathfrak{p}_0$.

Step 3: $\mathfrak{s} \subset \mathfrak{l}_0$, the Levi part of $\mathfrak{p}_0$

We write $\mathfrak{p}_0 = \mathfrak{n}_0 \rtimes \mathfrak{l}_0$ where $\mathfrak{n}_0$ is the unipotent radical of $\mathfrak{p}_0$. Note that $\mathfrak{l}_0$ is reductive in $\mathfrak{g}$ and hence $\mathfrak{h}$ is not contained in $\mathfrak{l}_0$. In addition we may assume that $\mathfrak{s} \subset \mathfrak{l}_0$ ([4] §6.8 Cor. 1).

Step 4: $\mathfrak{z}(\mathfrak{l}_0)$ is not contained in $\mathfrak{h}$

Since G/H is unimodular, $|\det \mathrm{Ad}(h)|_{\mathfrak{h}}| = 1$ for $h \in H$. If in addition $h \in Z(\mathfrak{l}_0)$, then h centralizes $\mathfrak{s}$ and it follows that $|\det \mathrm{Ad}(h)|_{\mathfrak{r}}| = 1$. Hence $\mathfrak{z}(\mathfrak{l}_0) \cap \mathfrak{h}$ centralizes $\mathfrak{r}$ as well. If $\mathfrak{z}(\mathfrak{l}_0)$ would be contained in $\mathfrak{h}$, and since $\mathfrak{l}_0$ is the centralizer of its center, then this would force $\mathfrak{h} \subset \mathfrak{l}_0$, contradicting Step 1.

Step 5: Decomposition $\mathfrak{p}_0 = \mathfrak{l}_1 \oplus (\mathfrak{h} + \mathfrak{n}_0)$.

Indeed, define the subspace $\mathfrak{l}_1 \subset \mathfrak{l}_0$ by

$$\mathfrak{l}_1 = \mathrm{pr}_{\mathfrak{l}_0}(\mathfrak{h})^{\perp}$$

where $\mathrm{pr}_{\mathfrak{l}_0} : \mathfrak{p}_0 \to \mathfrak{l}_0$ is the projection along $\mathfrak{n}_0$. Then $\mathfrak{l}_1 \oplus (\mathfrak{h} + \mathfrak{n}_0) = \mathfrak{p}_0$.

Step 6: The case $\mathfrak{p}_0 = \mathfrak{l}_1 \oplus \mathfrak{h} \oplus \mathfrak{n}_0$.

In this case $\mathfrak{h} \cap \mathfrak{n}_0 = \{0\}$ and thus the projection $\mathrm{pr}_{\mathfrak{l}_0}|_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{l}_0$, which is a Lie-algebra homomorphism, is injective. Write $\mathfrak{h}_0$ for the homomorphic image of $\mathfrak{h}$ in $\mathfrak{l}_0$. The analysis will be separated in two subcases.

Case 6a: $\mathfrak{h}_0$ is not reductive in $\mathfrak{l}_0$. Let H_0 and L_0 be the connected subgroups of G corresponding to $\mathfrak{h}_0$ and $\mathfrak{l}_0$. As G/H is unimodular and H is homomorphic to H_0, it follows that G/H_0 and thus L_0/H_0 is unimodular. By induction we find for every $1 \leq p < \infty$ an unbounded function $f \in L^p(L_0/H_0)^\infty$. As before in the case of $\tilde{H}/H$ we extend f to a smooth vector in $L^p(G/H)$ (note that $P_0/H \to L_0/H_0$ is a fibre bundle, and we first extend f to a function on P_0/H and then to a function on G/H).

Case 6b: $\mathfrak{h}_0$ is reductive in $\mathfrak{l}_0$. In particular it is a reductive Lie algebra, hence so is $\mathfrak{h}$. In the Levi decomposition $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{r}$ we now know that $\mathfrak{r}$ is the center of $\mathfrak{h}$. Let $\mathfrak{u}$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{r}$ and $\theta(\mathfrak{r})$, then $\mathfrak{s} + \mathfrak{u}$ is a direct Lie algebra sum. Moreover, $\mathfrak{s} + \mathfrak{u}$ is θ-invariant, hence reductive in $\mathfrak{g}$, and hence in fact $= \mathfrak{g}$ by our previous assumption on $\mathfrak{h}$. Thus $\mathfrak{s}$ is an ideal in $\mathfrak{g}$ which we may as well assume is 0. Now $\mathfrak{h} = \mathfrak{r}$ is an abelian subalgebra which together with $\theta(\mathfrak{r})$

generates $\mathfrak{g}$. We shall reduce to the case where $\mathfrak{r}$ is nilpotent in $\mathfrak{g}$, which we already treated in Section 5.1.

Every element $X \in \mathfrak{r}$ has a Jordan decomposition $X_n + X_s$ (in $\mathfrak{g}$), and we let $\mathfrak{o}_1, \mathfrak{o}_2$ be the subalgebras generated by the X_n's and X_s's, respectively. Then $\mathfrak{o} = \mathfrak{o}_1 \oplus \mathfrak{o}_2$ is abelian and $\mathfrak{o}_2$ consists of semisimple elements. The centralizer of $\mathfrak{o}_2$ is reductive in $\mathfrak{g}$ and contains $\mathfrak{r}$, hence equal to $\mathfrak{g}$. Hence $\mathfrak{o}_2$ is central in $\mathfrak{g}$, and we may assume that it is θ-stable. Let $\mathfrak{g}_1$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{o}_1$ and $\theta(\mathfrak{o}_1)$. It is reductive in $\mathfrak{g}$, and $(\mathfrak{g}_1, \mathfrak{o}_1)$ is of the type already treated, hence not VAI. Since $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{o}_2$ we can now conclude that $(\mathfrak{g}, \mathfrak{r}) = (\mathfrak{g}, \mathfrak{h})$ is not VAI either.

Step 7: An element $X \in \mathfrak{z}(\mathfrak{l}_0)$.

Using the result of Step 4, there exists $X \in \mathfrak{z}(\mathfrak{l}_0)\backslash\mathfrak{h}$ such that $\mathfrak{n}_0 \subset \mathfrak{g}_X^+$. As before we set $a_t := \exp(tX)$ and observe that $a_t z_0 \to \infty$ in Z for $|t| \to \infty$ (this is because $a_t[L_0, L_0]N_0$ tends to infinity in $G/[L_0, L_0]N_0$.)

Step 8: A decomposition $\mathfrak{p}_0 = \mathfrak{l}_1 \oplus \mathfrak{h} \oplus \mathfrak{n}_1$

We construct an $\operatorname{ad} X$-invariant subspace $\mathfrak{n}_1 \subset \mathfrak{n}_0$ such that $\mathfrak{h} + \mathfrak{n}_0 = \mathfrak{h} \oplus \mathfrak{n}_1$, as follows. If $\mathfrak{n}_0 \subset \mathfrak{h}$, then $\mathfrak{n}_1 = \{0\}$. Otherwise we choose an $\operatorname{ad} X$-eigenvector, say Y_1, in $\mathfrak{n}_0$ with largest possible eigenvalue, such that $\mathfrak{h} + \mathbb{R}Y_1$ is a direct sum. If this sum contains $\mathfrak{n}_0$, we set $\mathfrak{n}_1 = \mathbb{R}Y_1$. Otherwise we continue that procedure until a complementary subspace is reached. Now $\mathfrak{l}_1 \oplus \mathfrak{h} \oplus \mathfrak{n}_1 = \mathfrak{p}_0$ and by Step 6 we can assume $\mathfrak{n}_1 \subsetneq \mathfrak{n}_0$.

Step 9: We summarize the situation we have reduced to:

- $\mathfrak{h} = \mathfrak{r} \rtimes \mathfrak{s}$ is a Levi decomposition of $\mathfrak{h}$.
- $\mathfrak{h} \subset \mathfrak{p}_0 = \mathfrak{n}_0 \rtimes \mathfrak{l}_0$, a maximal parabolic subalgebra of $\mathfrak{g}$.
- $\mathfrak{s} \subset \mathfrak{l}_0$, the Levi part of $\mathfrak{p}_0$.
- $\mathfrak{l}_1 := \operatorname{pr}_{\mathfrak{l}_0}(\mathfrak{h})^\perp \subset \mathfrak{l}_0$ with $\operatorname{pr}_{\mathfrak{l}_0} : \mathfrak{p}_0 \to \mathfrak{l}_0$ the projection along $\mathfrak{n}_0$.
- $\mathfrak{p}_0 = \mathfrak{l}_1 \oplus \mathfrak{h} \oplus \mathfrak{n}_1$ with $\mathfrak{l}_1 \subset \mathfrak{l}_0$ and $\mathfrak{n}_1 \subsetneq \mathfrak{n}_0$.
- $X \in \mathfrak{z}(\mathfrak{l}_0)$ and
  1. $\mathfrak{n}_0 \subset \mathfrak{g}_X^+$.
  2. $\mathfrak{n}_1$ is invariant with respect to $\operatorname{ad}(X)$.
  3. With $a_t = \exp(tX)$ we have $a_t z_0 \to \infty$ in Z for $|t| \to \infty$.

We will construct (for any $1 \leq p < \infty$) a smooth function χ in $L^p(Z)$ which does not decay. For this we need some auxilary functions Φ_t which we now construct.

Let $\overline{\mathfrak{n}}_0$ be the nilradical of the parabolic opposite to $\mathfrak{p}_0$ and consider the $\operatorname{ad} X$-invariant vector space

$$\mathfrak{v} := \overline{\mathfrak{n}}_0 \times \mathfrak{l}_1 \times \mathfrak{n}_1 \subset \mathfrak{g}$$

which is complementary to $\mathfrak{h}$. For fixed $t \in \mathbb{R}$ we define the differentiable map

$$\Phi = \Phi_t : \mathfrak{v} \to Z,$$

by the formula

$$\Phi(Y^-, Y^0, Y^+) = \exp(Y^-)\exp(Y^0)\exp(Y^+)a_t z_0 \, .$$

The main property which we need of these functions is expressed in the following lemma. For $Y = (Y^-, Y^0, Y^+) \in \mathfrak{v}$ we put

$$y^{\pm,0} = \exp(Y^{\pm,0}) \in G$$

and $y = y^- y^0 y^+$, and we identify the tangent space $T_{\Phi_t(Y)}Z$ with $\mathfrak{v}$ via the map

$$T_{\Phi_t(Y)}Z \to \mathfrak{v}, \quad d\tau_{ya_t}(z_0)(X + \mathfrak{h}) \mapsto \pi_{\mathfrak{v}}(X + \mathfrak{h}), \quad (X \in \mathfrak{g})$$

where $\pi_{\mathfrak{v}} : \mathfrak{g} \to \mathfrak{v}$ is the projection along $\mathfrak{h}$.

Lemma 5.4. *For every sufficiently small compact neighborhood Q of 0 in $\mathfrak{v}$, there exist constants $c_Q, C_Q > 0$ such that*

$$c_Q e^{t\gamma} \leq \sup_{Y \in Q} |\det d\Phi_t(Y)| \leq C_Q e^{t\gamma} \qquad (t \leq 0).$$

In particular $\Phi_t|_Q$ is a chart for all $t \leq 0$.

The proof, which is computational, is postponed to the end of this section. The construction of the function χ is now easy to describe. Let $Q \subset \mathfrak{v}$ be as above. We fix a function $\psi \in C_c^\infty(Q)$ with $0 \leq \psi \leq 1$ and $\psi(0) = 1$. For all $t < 0$ define $\chi_t \in C_c^\infty(Z)$ by $\chi_t(z) = \psi(\Phi_t^{-1}(z))$ and set

$$\chi := \sum_{n \in \mathbb{N}} n\chi_{-n} \, .$$

It is clear that $\chi \in C^\infty(Z)$ and that χ is unbounded. We claim that $\chi \in L^p(Z)^\infty$.

It follows immediately from the definition that $\chi_t \in L^p(Z)$ for all $1 \leq p < \infty$ and $t \leq 0$, and it follows from the estimate of the differential of Φ in Lemma 5.4 that $\|\chi_t\|_p \leq Ce^{t\gamma/p}$ for some $C > 0$ not depending on t (but possibly on p). Hence

$$\chi = \sum_{n \in \mathbb{N}} n\chi_{-n} \in L^p(Z)$$

for all $1 \leq p < \infty$, and it only remains to be seen that also the derivatives of χ belong to $L^p(Z)$. The proof of this fact depends in addition on the following estimate, which will be proved together with Lemma 5.4.

Lemma 5.5. *Define*

$$M_t := \sup_{U\in\mathfrak{g}, \|U\|=1} \| \operatorname{Ad}(a_t)\pi_{\mathfrak{v}} \operatorname{Ad}(a_t)^{-1}U\|$$

Then $\sup_{t<0}(M_t) < \infty$.

We now complete the proof of the Proposition 5.1 by proving that the left derivatives of χ by elements $U \in \mathfrak{g}$, up to all orders, belong to $L^p(Z)$.

We first show this for first order derivatives. Let $U \in \mathfrak{g}$ and consider the derivative $L(U)\chi_t$. At $z = \Phi_t(Y)$ this is given by

$$L(U)\chi_t(z) = d/ds|_{s=0}\, \chi_t(\exp(sU)ya_tz_0).$$

For Y in a compact set, we can replace U by its conjugate by y without loss of generality, and thus we may as well consider the derivatives of

$$\chi_t(y\exp(sU)a_tz_0).$$

We rewrite this as

$$\chi_t(ya_t\exp(s\operatorname{Ad}(a_t)^{-1}U)z_0)$$

and apply the projection along $\mathfrak{h}$. It follows that the derivative can be rewritten as

$$d/ds|_{s=0}\, \chi_t(ya_t\exp(s\pi_{\mathfrak{v}}\operatorname{Ad}(a_t)^{-1}U)z_0)$$

and then finally also as

$$d/ds|_{s=0}\, \chi_t(y\exp(s\operatorname{Ad}(a_t)\pi_{\mathfrak{v}}\operatorname{Ad}(a_t)^{-1}U)a_tz_0).$$

Note that $\operatorname{Ad}(a_t)\pi_{\mathfrak{v}}\operatorname{Ad}(a_t)^{-1}U \in \mathfrak{v}$. We conclude that the derivative is a linear combination of derivatives of ψ on Q, with coefficients that depend smoothly on Y. Furthermore, it follows from Lemma 5.5 that the coefficients are bounded for $t < 0$. As before we conclude $L(U)\chi_t \in L^p(Z)$ for all $t \le 0$, with exponentially decaying p-norms. It follows that $L(U)\chi \in L^p(Z)$.

By repeating the argument for higher derivatives we finally see that $\chi \in L^p(Z)^\infty$. □

It remains to verify Lemmas 5.4 and 5.5. We first prove the latter.

Proof of Lemma 5.5. For $U \in \mathfrak{v}$ we have

$$\operatorname{Ad}(a_t)\pi_{\mathfrak{v}}\operatorname{Ad}(a_t)^{-1}U = U,$$

hence we may assume $U \in \mathfrak{h}$. Since $\mathfrak{h} \subset \mathfrak{p}_0$ we can write U as a combination of an element $Y_0 \in \mathfrak{l}_0$ and possibly some $\operatorname{ad} X$-eigenvectors Y_λ with eigenvalues $\lambda > 0$. Then

$$\operatorname{Ad}(a_t)^{-1}U = Y_0 + \sum e^{-\lambda t}Y_\lambda = U + \sum(e^{-\lambda t} - 1)Y_\lambda$$

(possibly with an empty sum). If $Y_\lambda \in \mathfrak{n}_1$ then

$$\operatorname{Ad}(a_t)\pi_{\mathfrak{v}}(e^{-\lambda t}-1)Y_\lambda = (1-e^{\lambda t})Y_\lambda \to Y_\lambda$$

as $t \to -\infty$. On the other hand it follows from the definition of $\mathfrak{n}_1$, that if Y_λ is not in $\mathfrak{n}_1$ then either it belongs to $\mathfrak{h}$ or it is a sum of an element from $\mathfrak{h}$ and some eigenvectors $V_\mu \in \mathfrak{n}_1$ with eigenvalues $\mu \geq \lambda$. Then

$$\operatorname{Ad}(a_t)\pi_{\mathfrak{v}}(e^{-\lambda t}-1)Y_\lambda = \sum e^{\mu t}(e^{-\lambda t}-1)V_\mu$$

(possibly with an empty sum), which stays bounded for $t \to -\infty$. Our claim is thus established. □

To prepare the proof of Lemma 5.4 we establish the following lemma. To simplify the main formula below we denote

$$\beta(T) = \frac{\mathbf{1} - e^{-\operatorname{ad} T}}{\operatorname{ad} T} \in \operatorname{End}(\mathfrak{g})$$

for $T \in \mathfrak{g}$. Note that $\beta(0) = \mathbf{1}$.

Lemma 5.6. *Let $Y = (Y^-, Y^0, Y^+) \in \mathfrak{v}$.*

(1) Let $X = (X^-, X^0, X^+) \in \mathfrak{v}$, then $d\Phi_t(Y)(X) \in \mathfrak{v}$ is given by

$$d\Phi_t(Y)(X) = \pi_{\mathfrak{v}} \circ \operatorname{Ad}(a_t)^{-1}(S_{Y,X})$$

where $S_{Y,X} \in \mathfrak{g}$ is the element

$$\operatorname{Ad}(y_0 y^+)^{-1}\beta(Y^-)(X^-) + \operatorname{Ad}(y^+)^{-1}\beta(Y^0)(X^0) + \beta(Y^+)(X^+).$$

(2) There exists a linear map $L(Y) : \mathfrak{v} \to \mathfrak{g}$ such that

$$d\Phi_t(Y) = \operatorname{Ad}(a_t)^{-1}(\mathbf{1}_{\mathfrak{v}} + \operatorname{Ad}(a_t)\pi_{\mathfrak{v}}\operatorname{Ad}(a_t)^{-1}L(Y))$$

for all $t \leq 0$, and such that $\|L(Y)\| \to 0$ for $Y \to 0$.

Proof. We get for the differential of Φ:

$$d\Phi(Y^-, Y^0, Y^+)(X^-, X^0, X^+) = d\tau_{y^- y^0 y^+ a_t}(z_0) \circ \operatorname{Ad}(a_t)^{-1}(S_{Y,X})$$

with $S_{Y,X}$ as above. Using the identification of the tangent space with $\mathfrak{v}$ this is exactly the statement of item (1).

Defining $L(Y)$ by $L(Y)(X) = S_{Y,X} - X$ for $X \in \mathfrak{v}$, we obtain the expression in item (2). It is easily seen that $\|L(Y)\| \to 0$ for $Y \to 0$. □

Proof of Lemma 5.4. Finally, it follows from Lemma 5.5 that $\operatorname{Ad}(a_t)^{-1}\mathbf{1}_{\mathfrak{v}}$ dominates in the expression in item (2) above, for $Y \in \mathfrak{v}$ sufficiently small. Since $\mathfrak{n}_1$ is proper in $\mathfrak{n}$, we obtain Lemma 5.4. □

5.3. **Final remarks.** **1.** We did not address here the case where G is not reductive. One might expect for G and H algebraic and G general, that Z has VAI if and only if the nilradical of H is contained in the nilradical of G.

2. The following may be an alternative approach to Theorem 1.3 for algebraic groups G and H. To be more specific, assume G and $H < G$ to be complex algebraic groups and $Z = G/H$ to be unimodular and quasi-affine. Under these assumptions we expect that there is a rational G-module V, and an embedding $Z \to V$ such that the invariant measure μ_Z, via pull-back, defines a tempered distribution on V. Note that if Z is of reductive type, then there exists a V such that the image of $Z \to V$ is closed, and hence μ_Z defines a tempered distribution on V. If Z is not of reductive type, then by Matsushima's criterion ([3], Thm. 3.5) all images $Z \to V$ are non-closed and the expected embedding would imply that VAI does not hold. This is supported by a result of Deligne, established in [14], which asserts that for a reductive group G and $X \in \mathfrak{g} := \mathrm{Lie}(G)$ the invariant measure on the adjoint orbit $Z := \mathrm{Ad}(G)(X) \subset \mathfrak{g}$ defines a tempered distribution on $\mathfrak{g}$. Various particular results in the theory of prehomogeneous vector spaces provide additional support (see [1]).

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